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On the Number of Local Best Approximations by Exponential Sums

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Let E_n denote the set of exponential sums of degree $\leqslant n$. According to D. Braess the maximal number of local best approximations in E_n, $n \geqslant 4$, is at most $\frac{3}{2}(n-1)!$. It is proven that the asymptotic growth of this bound is correct by constructing functions which have at least $\max\{\frac{1}{2}n^2 - \frac{3}{2}n + 3,\ 2^{(2/3)n}/5\sqrt{n}\}$ different local solutions in E_n, $n \geqslant 4$. Moreover, examples are given of functions with at least six local solutions in E_4 and exactly three global solutions in E_3.

INTRODUCTION

This paper is concerned with the uniform approximation of continuous functions on the interval $I = [-1, 1]$ by exponential sums of degree $\leqslant n$, i.e., by elements of the space

$$E_n := \left\{ u(x) = \sum_{i=1}^{l} \sum_{j=0}^{m_i - 1} a_{ij} x^j e^{t_i x} : \right.$$
$$\left. l, m_i \in \mathbb{N}, a_{ij}, t_i \in \mathbb{R}, t_1 < \cdots < t_l, \sum_{i=1}^{l} m_i \leqslant n \right\}. \qquad (0.1)$$

It is known from [1] that for $n \geqslant 2$ best approximation (BA) may not be unique and that there may also exist local best approximations (LBAs). This gives rise to the question whether the number of LBAs is always finite and, if the answer is affirmative, what is the value of the number

$$c_n := \sup_{f \in C(I)} \{\text{number of LBAs to } f \text{ in } E_n\}. \qquad (0.2)$$

It is known from [2] and [6] that

$$c_n = n, \qquad n = 1, 2, 3,$$

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and

$$n \leqslant c_n \leqslant \tfrac{3}{2}(n-1)!, \qquad n \geqslant 4.$$

In this paper we will partially fill this gap by proving the estimates

$$\begin{aligned} c_n &\geqslant 2 + \tfrac{1}{2}(n-1)(n-2) \\ &\geqslant \max_{1 \leqslant i \leqslant [(n-1)/2]} \binom{n-1-i}{i}, \qquad n \geqslant 4. \end{aligned}$$

The proof is by constructing special functions with the given number of LBAs. The first estimate is good for small values of n, whereas the second one shows that c_n grows faster than any polynomial of n when n tends to infinity.

In the case $n = 4$ the above estimates can be improved:

$$6 \leqslant c_4 \leqslant 9.$$

Finally, we show the existence of a function with three global BAs in E_3, thus answering another open question (cf. [5]).

1. Preliminaries

The sign vectors of exponential sums (cf. Section 3 in [2]) play an important role in our investigations. Consider an exponential sum

$$u(x) = \sum_{i=0}^{m-1} a_i x^i e^{tx}, \qquad a_{m-1} \neq 0,$$

with only one characteristic number t. Let $s = \operatorname{sign} a_{m-1}$. Then the sign vector sign $(u) \in \{-1, 1\}^m$ is given by

$$\operatorname{sign}(u) := ((-1)^{m-1} s, (-1)^{m-2} s, \ldots, -s, s).$$

Now let u_1, u_2 be two exponential sums such that all characteristic numbers of u_1 are smaller than each of u_2. Then the sign vector of $u_1 + u_2$ is given by

$$\operatorname{sign}(u_1 + u_2) := (\operatorname{sign}(u_1), \operatorname{sign}(u_2)).$$

Each vector $s \in \{-1, 1\}^n$ specifies a sign class $E_n(s)$ of E_n:

$$E_n(s) := \{u \in E_n \backslash E_{n-1} : \operatorname{sign}(u) = s\}.$$

Moreover, E_n^+ denotes the set of all positive exponential sums of degree $\leqslant n$

(cf. Section 5 in [2]). With the help of the sign vectors one can formulate a Descartes rule (cf. Theorem 3.2 in [2]).

LEMMA 1.1. (Descartes rule). *Let* $u \in E_k$, $k \in \mathbb{N}$, *satisfy*

$$(-1)^{i-1} u(x_i) \geqslant 0, \qquad i = 1, 2,..., r,$$

with $x_1 < x_2 < \cdots < x_r$ *and* $r \geqslant 2$. *If* $u \neq 0$ *and* sign $(u) = (s_1,..., s_k)$, *then the sequence* $s_1,..., s_k$ *contains at least* $r-1$ *sign changes. Moreover, if the number of sign changes is exactly* $r-1$, *we have*

$$s_1 = 1 \quad \textit{and} \quad s_k = (-1)^{r-1}.$$

We will frequently use the Descartes rule in the following form.

Remark 1.2. Let $f \in C(I)$ and $u \in E_n$. Assume that $f-u$ has an alternant of length k with sign σ on the right

$$\sigma(-1)^{k-i}(f(x_i) - u(x_i)) = \|f-u\|, \qquad i = 1, 2,..., k.$$

Furthermore, let $v \in E_m$ be a better approximation to f than u. Then sign $(v-u)$ contains at least $k-1$ sign changes. If the number of sign changes is exactly $k-1$, then the first and last component of sign$(v-u)$ equal $(-1)^{k-1}\sigma$ and σ.

Another application of the Descartes rule is the following lemma (cf. Theorems 4.3, 4.5 in [2]).

LEMMA 1.3. *Let* $u \neq 0$ *be a best approximation to* $f \in C(I)$ *in* E_n *with simple characteristic numbers. Assume that the alternant of* $f-u$ *has a positive (negative) sign on the right. If* v *is an approximation to* f *at least as good as* u, *then* sign(v) *contains at least as many negative (positive) components as* sign(u) *and at least one more positive (negative) component than* sign(u).

Another central concept is the standard construction of D. Braess (cf. Section 18 in [6]). Beginning with a known LBA to f in E_{n-1}, $n \geqslant 2$, it achieves a connected set Cp of E_n which contains exactly one LBA to f in E_n whenever $Cp \cap E_{n-1} = \varnothing$. Given $\tau \in \mathbb{R}$, let

$$E_n(\tau) := \{u \in E_n : \tau \in \text{spect}(u) \text{ or } u \in E_{n-1}\}$$

be the closure of the subset of exponential sums of degree $\leqslant n$ with characteristic number τ. Here, spect (u) denotes the set of characteristic numbers of

u. Put $E_0 := \{0\}$. Let $u \in E_{n-1} \backslash E_{n-2}$, $n \geqslant 2$, and $\tau \notin \operatorname{spect}(u)$. Then the tangent cone to $E_n(\tau)$ at u is given by

$$C_u E_n(\tau) = C_u E_{n-1} + \{\alpha e^{\tau x} : \alpha \in \mathbb{R}\},$$

where $C_u E_{n-1}$ is the tangent cone to E_{n-1} at u (cf. [6]). The standard construction now reads as follows.

Standard construction. Let $u \in E_{n-1} \backslash E_{n-2}$, $n \geqslant 2$, be a local best approximation in E_{n-1} and $\tau \notin \operatorname{spect}(u)$. Assume that $f-u$ has an alternant of exact length $n+l$, where l is the number of different characteristic numbers of u. Let $h \in C_u E_n(\tau)$ be a function with

$$\|f-u-h\| < \|f-u\|$$

and $\{u_\lambda : \lambda \in [0, 1]\}$ be a continuous curve in $E_n(\tau)$ with

$$\|u_\lambda - u - \lambda h\| = o(\lambda) \qquad \text{as} \quad \lambda \to 0. \tag{1.1}$$

Then define Cp as the connected component of the level set $\{v \in E_n : \|f-v\| < \|f-u\|\}$ that contains the curve u_λ for sufficiently small $\lambda > 0$.

The component Cp is well defined and depends only on u and on the connected component of $\mathbb{R} \backslash \operatorname{spect}(u)$ that contains τ (cf. Section 18 in [6]). If Cp does not intersect E_{n-1}, it contains exactly one LBA to f in E_n. (Otherwise it may be discarded, when all local solutions in E_n are to be determined.) We are mostly interested in the sign vector of this LBA. Write u according to (0.1)

$$u(x) = \sum_{i=1}^{l} \sum_{j=0}^{m_i - 1} a_{ij} x^j e^{t_i x}, \qquad x \in I,$$

with $\sum_{i=1}^{l} m_i = n-1$, $a_{i m_i - 1} \neq 0$, $i = 1, \ldots, l$, and $t_1 < \cdots < t_l$. Furthermore, put

$$t_0 := -\infty, \qquad t_{l+1} := \infty$$

and let

$$\operatorname{sign}(u) = (s_1, \ldots, s_{n-1}).$$

Then we can prove the following:

THEOREM 1.4. *Suppose that u satisfies the conditions for the standard construction. Let $i \in \{0, 1, \ldots, l\}$ and $t_i < \tau < t_{i+1}$. Referring to* (0.1) *set*

$$j(i) := 1 + \sum_{k=1}^{i} m_k$$

and let Cp be the connected set of E_n which is given by the standard construction applied to u with additional characteristic number τ. Then Cp contains an element $\tilde{u} \in E_n \backslash E_{n-1}$ whose sign vector sign $(\tilde{u}) = (\tilde{s}_1, \ldots, \tilde{s}_n)$ is given by

$$\begin{aligned} \tilde{s}_k &= s_k, && 1 \leqslant k < j(i), \\ &= \sigma(-1)^{n+l-j(i)-i}, && k = j(i), \\ &= s_{k-1}, && j(i) < k \leqslant n, \end{aligned} \tag{1.2}$$

where σ is the right sign of the alternant of $f-u$.

Proof. Since $f-u$ has an alternant of exact length $n+l$, the BA h to $f-u$ in the $n+l$ dimensional Haar subspace

$$W := \left\{ \sum_{\nu=1}^{l} \sum_{\mu=0}^{m_\nu} \beta_{\nu\mu} x^\mu e^{t_\nu x} + \delta e^{\tau x} : \beta_{\nu\mu}, \delta \in \mathbb{R} \right\}$$

of $C_u E_n(\tau)$ has the representation $h = \tilde{h} + \alpha e^{\tau x}$ with $\tilde{h} \in C_u E_{n-1}$ and $\alpha \neq 0$.

From Remark 1.2 and $h \in E_{n+l}$ we know that sign(h) contains exactly $n+l-1$ sign changes and that the last component equals σ. Since τ is the $j(i)+i$th characteristic number of h, this yields

$$\operatorname{sgn} \alpha = \sigma(-1)^{n+l-j(i)-i}.$$

Since $\tilde{h} \in C_u E_{n-1}$ and $u \in E_{n-1} \backslash E_{n-2}$, there is a continuous curve $\{\tilde{u}_\lambda : \lambda \in [0, 1]\}$ in $E_{n-1} \backslash E_{n-2}$ with

$$\|\tilde{u}_\lambda - u - \lambda \tilde{h}\| = o(\lambda) \qquad \text{as} \quad \lambda \to 0. \tag{1.3}$$

Because of the continuity we may assume that τ lies between the ith and $i+1$st characteristic number of $\tilde{u}_\lambda$ and that

$$\operatorname{sign}(\tilde{u}_\lambda) = \operatorname{sign}(u), \qquad \lambda \in [0, 1].$$

This implies that the sign vector of

$$u_\lambda := \tilde{u}_\lambda + \alpha \lambda e^{\tau x}, \qquad \lambda \in [0, 1],$$

satisfies (1.2) for every $\lambda \in (0, 1]$. Moreover, (1.3) implies that $\{u_\lambda : \lambda \in [0, 1]\}$ satisfies (1.1). Hence, $u_\lambda \in Cp$ for sufficiently small $\lambda > 0$. ∎

2. A Quadratic Lower Bound for c_n

We introduce some families of functions we need for the proof of our estimates. Let $m \geqslant 2$ and $a \in \mathbb{R}$. Then we put

$$g_m(x) := \sum_{i=1}^{m} \exp\left(\left(-1 + 2\frac{i-1}{m-1}\right)x\right) \in E_m^+, \qquad x \in I = [-1, 1], \tag{2.1}$$

and

$$f_m(a; x) := (-1)^{m-1} \cos \frac{2m+1}{2} \pi x + a g_m(x), \qquad x \in I. \tag{2.2}$$

If no confusion may arise, we simply write f_m instead of $f_m(a; .)$.

The choice of the functions $f_m(a; .)$ is rather arbitrary. We will only refer to the following two properties.

(i) $f_m - a g_m$ is an even function with a negative alternant of exact length $2m+1$.

(ii) g_m is an even positive exponential sum of exact degree m.

Next, we need some sign classes. For ease of notation we denote by $s_{j,n} \in \{-1, 1\}^n$, $1 \leqslant j \leqslant n$, the sign vector with $n-1$ positive components whose jth component equals -1.

Definition 2.1. Let $S_{m,n}$, $2 \leqslant m < n \leqslant 2m-1$, denote the set of all vectors $s = (s_1, ..., s_n) \in \{-1, 1\}^n$ with the following properties.

(i) m components of s equal $+1$ and $n-m$ components equal -1.

(ii) $s_j = -1$ implies $s_{j+1} = +1$, $1 \leqslant j \leqslant n-1$.

(iii) $s_1 = s_n = +1$.

Definition 2.2. (a) For $m \geqslant 2$ let

$$\hat{S}_{m,m+1} := S_{m,m+1} \cup \{s_{1,m+1}, s_{m+1,m+1}\}.$$

(b) Let $\hat{S}_{m,m+2}$, $m \geqslant 2$, denote the set of all vectors $s \in \{-1, 1\}^{m+2}$ which equal $s_{2,m+2}$ or $s_{m+1,m+2}$ or satisfy conditions (i) and (ii) of Definition 2.1 for $S_{m,m+2}$.

Now we can do the first step in the proof of the quadratic lower bound for c_n.

Lemma 2.3. *Let $m \geqslant 2$, $N := m+1$ and $a > 0$. Each sign class $E_N(s)$, $s \in \hat{S}_{m,N}$, contains a local best approximation $u_{N,s} = u_{N,s}(a; .)$ to $f_m(a; .)$ in E_N which satisfies the following conditions:*

(i) *$u_{N,s}$ is a best approximation to f_n in the closure $\overline{E_N(s)}$ of $E_n(s)$.*

(ii) $\operatorname{spect}(u_{N,s}) \subset [-1, 1]$ *for every $s \in S_{m,N}$.*

(iii) *There is a number $\varepsilon_1 > 0$ such that the error curve $f_m(a; .) - u_{N,s}(a; .)$ has a negative alternant of exact length $2m + 1$ for every $a \in (0, \varepsilon_1)$ and $s \in S_{m,N}$. Moreover, $u_{N,s}(a; .)$ has $N - 3$ characteristic numbers with multiplicity one and one with multiplicity three.*

Proof. Since $f_m(a; \cdot) - ag_m$ has a negative alternant of exact length $2m + 1$, we conclude from Theorem 6.1 in [2] that $ag_m \in E_m^+$ is the unique BA to $f_m(a; \cdot)$ in E_m and satisfies the assumptions of Theorem 1.4 with $\sigma = -1$. Let $s \in \hat{S}_{m,N}$. Then $s = s_{j_0,N}$ for a suitable $j_0 \in \{1,..., N\}$. Let $t_1 < \cdots < t_m$ be the characteristic numbers of ag_m and

$$t_0 := -\infty, \qquad t_{m+1} := +\infty.$$

When applying the standard construction to ag_m with an additional characteristic number $\tau \in (t_{j_0-1}, t_{j_0})$, by Theorem 1.4 we obtain a connected component C of the level set $\{v \in E_N : \|f_m - v\| < \|f_m - ag_m\|\}$ which intersects $E_N(s)$. Since ag_m is the BA to $f_m(a; \cdot)$ in E_m, we have

$$C \cap E_{N-1} = C \cap E_m = \varnothing.$$

From Satz 2 in [8] it is known that there exists a BA $u_{N,s}(a; \cdot)$ to $f_m(a; \cdot)$ in $\overline{E_N(s)}$. Since $C \subset E_N(s)$ already contains elements which are better approximations than the best one in E_{N-1}, it follows from $\overline{E_N(s)} \subset E_N(s) \cup E_{N-1}$ that $u_{N,s}(a; \cdot) \in E_N(s)$. Hence, $u_{N,s}(a; \cdot)$ is an LBA. Especially, we have

$$\|f_m(a; \cdot) - u_{N,s}(a; \cdot)\| < \|f_m(a; \cdot) - ag_m\|. \tag{2.3}$$

For the proof of (ii) let $s \in S_{m,N}$. Then sign $(u_{N,s}(a; \cdot) - ag_m)$ contains at most $m = N - 1$ positive components. Therefore, from (2.3) and Remark 1.2 it follows that sign $(u_{N,s}(a; \cdot) - ag_m)$ contains exactly $2m$ sign changes and that the first and last component are both negative. Hence the origin of these two components is $-ag_m$ and not $u_{N,s}(a; \cdot)$. This yields

$$\operatorname{spect}\,(u_{N,s}(a; \cdot)) \subset [t_1, t_m] = [-1, 1].$$

For the proof of (iii) we define

$$u_{N,s}(0; x) := 0, \qquad x \in I, s \in S_{m,N},$$

and prove that the function

$$\phi\colon [0, \infty) \to C(I),$$
$$\phi(a) := f_m(a; \cdot) - u_{N,s}(a; \cdot)$$

is continuous at $a = 0$ for every $s \in S_{m,N}$. (It is no drawback that $u_{N,s}(a; \cdot)$ is not always unique.) Let $s \in S_{m,N}$ and $(a_\nu) \subset (0, \infty)$ be an arbitrary sequence with $\lim_{\nu \to \infty} a_\nu = 0$. By standard arguments (2.3) implies the boundedness of $(\| u_{N,s}(a_\nu; \cdot)\|)$. Since spect $(u_{N,s}(a_\nu; \cdot))$ is contained in the compact interval $[-1, 1]$ for all $\nu \in \mathbb{N}$ we conclude with the same arguments as in [7] that a subsequence of $(u_{N,s}(a_\nu; \cdot))$ converges uniformly on I to a function $v \in E_N$. From (2.3), we obtain

$$\| f_m(0; \cdot) - v \| \leqslant \| f_m(0; \cdot)\|.$$

This implies $v = 0$ because zero is the unique BA to $f_m(0; \cdot)$ in E_k for every $k \leqslant 2m$. Now the continuity of ϕ is obvious. The continuity of ϕ at $a = 0$ for every $s \in S_{m,N}$ implies the existence of a number $\varepsilon_1 > 0$ such that the error curve $f_m(a; \cdot) - u_{N,s}(a; \cdot)$ has an alternant of length $2m + 1$ at most for every $a \in (0, \varepsilon_1)$ and $s \in S_{m,N}$ because this is true for $a = 0$. By the same argument, this alternant has a negative sign on the right if it has the exact length $2m + 1$. By Theorem 6.1 in [2], $u_{N,s}(a; \cdot)$ has at most $m - 1$ distinct characteristic numbers. Since s contains exactly one negative component, $u_{N,s}(a; \cdot)$, however, has at least $m - 1$ different characteristic numbers. This implies assertion (iii) of the lemma. ∎

The next step in the proof of the quadratic lower bound for c_n is the following:

LEMMA 2.4. *Let $m \geqslant 2$, $n := m + 2$ and $a \in (0, \varepsilon_1)$, where ε_1 is chosen according to Lemma 2.3. Then each sign class $E_n(s)$, $s \in \hat{S}_{m,n}$, contains at least one local best approximation to $f_m(a; \cdot)$ in E_n.*

Proof. Let $s^{(1)} := s_{1,m+1}$ and $s^{(2)} := s_{m+1,m+1}$ be the two elements of $\hat{S}_{m,m+1} \backslash S_{m,m+1}$. First, we prove that the function $u_{n-1,s^{(1)}}$ satisfies the assumptions of Theorem 1.4 with $\sigma = -1$. The function $u_{n-1,s^{(1)}}$ cannot be even and has at least $m = n - 2$ different characteristic numbers. This and the symmetry of f_m imply that $u_{n-1,s^{(1)}}$ is not the unique BA. From Theorem 6.1 in [2] we conclude that $f_m - u_{n-1,s^{(1)}}$ has an alternant of exact length $2n - 2$. The assertion concerning σ now follows immediately from the definition of $s^{(1)}$ and Theorem 12.3 in [3]. With the same arguments we can prove that $u_{n-1,s^{(2)}}$ satisfies the assymptions of Theorem 1.4 with $\sigma = +1$.

Now, let t_0 be the smallest characteristic number of $u_{n-1,s^{(1)}}$ and $\tau < \min\{t_0, -1\}$. When applying the standard construction to $u_{n-1,s^{(1)}}$ with additional characteristic number τ, we obtain a connected component C of the level set $\{v \in E_n : \| f_m - v \| < \| f_m - u_{n-1,s^{(1)}} \| \}$, which intersects $E_n(s_{2,n})$. Let A denote the set of the elements of $E_n(s_{2,n})$ with first characteristic number smaller than -1. By construction

$$C \cap A \neq \varnothing. \tag{2.4}$$

Next, we want to prove

$$C \cap \partial A = \varnothing. \tag{2.5}$$

Assume that $v \in C \cap \partial A$. Denote the smallest characteristic number of v by t. Let $(v_n) \subset C \cap A$ be a sequence which converges uniformly on every compact subset of $(-1, 1)$ to v and let t_n denote the smallest characteristic number of v_n. We consider two cases.

Case 1. $v \in E_n \backslash E_{n-1}$. From Theorem 8.3 in [2] we know that $\lim_{n\to\infty} t_n = t$. This and $v \in \partial A$ imply $t = -1$. Thus we have $v - ag_m \in E_{2m+1}$. Moreover, we know from $ag_m \in E_m^+$ and $v \in E_n(s_{2,n})$ that the second component of sign $(v - ag_m)$ is negative. Since $v \in C$ is a better approximation to f_m than ag_m, by Remark 1.2 this contradicts that $f_m - ag_m$ has a negative alternant of length $2m + 1$.

Case 2. $v \in E_{n-1}$. First we recall the following cancellation rule for the boundary of a sign class

$$\partial E_n(s_1, \ldots, s_n) \subset E_{n-2} \cup \bigcup_{j=1}^{n} E_{n-1}(s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_n). \tag{2.6}$$

Especially we have

$$v \in \partial E_n(s_{2,n}) \subset E_{n-1}(s_{1,n-1}) \cup E_{n-1}(s_{2,n-1}) \cup E_{n-1}^+ \cup E_{n-2}.$$

Moreover, v is a better approximation to f_m than ag_m and $u_{n-1,s^{(1)}}$. We conclude from Lemma 1.3 that $v \notin E_{n-1}^+ \cup E_{n-2}$ and from Lemma 2.3 that $v \notin \overline{E_{n-1}(s_{1,n-1})}$. Thus, we have $v \in E_{n-1}(s_{2,n-1})$ and a subsequence of (t_n) converges to t. Hence $t \leqslant -1$. With the same arguments as in the proof of Lemma 2.3(ii) we conclude from $v \in E_{n-1}(s_{2,n-1})$ that $t \geqslant -1$ and $v - ag_m \in E_{2m+1} \backslash E_{2m}$. Hence $t = -1$. Because of (2.1) and $v \in E_{m+1}$ this is a contradiction to $v - ag_m \in E_{2m+1} \backslash E_{2m}$.

Thus (2.5) is proved. Since A is an open subset of E_n and C is connected, (2.4) and (2.5) imply

$$C \subset A \subset E_n \backslash E_{n-1}.$$

Hence, C contains an LBA u to f_m in E_n and sign $(u) = s_{2,n}$. Since f_m is an even function, $v(x) := u(-x)$ is also an LBA to f_m in E_n. Obviously, we have sign $(v) = s_{n-1,n}$. To complete the proof of Lemma 2.4, we have to prove the existence of at least one LBA to f_m in each sign class $E_n(s)$ with $s \in \hat{S}_{m,n} \backslash \{s_{2,n}, s_{n-1,n}\}$. Let $E_n(s)$ be such a sign class. Then s contains exactly two negative components. Cancelling one of them we obtain two elements σ, σ' of $\hat{S}_{m,n-1}$. Without loss of generality we may assume

$$\| f_m - u_{n-1,\sigma} \| \leqslant \| f_m - u_{n-1,\sigma'} \|. \tag{2.7}$$

σ may be obtained from s by canceling the j_0th component. Define the zeroth characteristic number of $u_{n-1,\sigma}$ as $-\infty$ and the nth one as $+\infty$. We apply the standard construction to $u_{n-1,\sigma}$ with an additional characteristic number between the j_0-1st and j_0th one. (Note that we now count the characteristic numbers according to their multiplicity.) Doing this we obtain a connected component C of the set of all elements v in $E_n(s)$ satisfying

$$\|f_m - v\| < \|f_m - u_{n-1,\sigma}\|. \tag{2.8}$$

If $v \in C \cap \partial E_n(s)$, we have according to (2.6)

$$v \in E_{n-1}(\sigma) \cup E_{n-1}(\sigma') \cup E_{n-2}^{+} \cup \{u \in E_{n-1}\colon \text{sign } (u) \text{ contains at most } m-1 \text{ positive components}\}. \tag{2.9}$$

Since v is a better approximation to f_m than ag_m, we conclude from (2.8) and Lemma 1.3 that $v \in E_{n-1}(\sigma) \cup E_{n-1}(\sigma')$. This contradicts Lemma 2.3 because of (2.7) and (2.8). Hence, $C \cap \partial E_n(s) = \varnothing$ and C does not intersect E_{n-1}. This implies the existence of an LBA to f_m in $E_n(s)$. ∎

Lemma 2.4 yields the inequality

$$\begin{aligned} c_{m+2} &\geqslant \#\hat{S}_{m,m+2} \\ &= 2 + \#(\hat{S}_{m,m+2} \backslash \{s_{2,n}, s_{n-1,n}\}) \\ &= 2 + \tfrac{1}{2}m \cdot (m+1), \qquad m \geqslant 2. \end{aligned}$$

As a consequence we get:

THEOREM 2.5. *The maximal number of local best approximations in* E_n, $n \geqslant 4$, *is at least*

$$c_n \geqslant 2 + \tfrac{1}{2}(n-1)(n-2). \tag{2.10}$$

3. AN EXPONENTIAL LOWER BOUND FOR c_n

In this section we prove the following lemma which has an exponentially increasing lower bound for c_n as an immediate consequence.

LEMMA 3.1. *Let* $2 \leqslant m < n \leqslant 2m-1$. *There is a number* $\varepsilon_n > 0$ *such that for every* $a \in (0, \varepsilon_n)$ *and* $s \in S_{m,n}$ *the function* $f_m(a; \cdot)$ *has a local best approximation* $u_{n,s} = u_{n,s}(a; \cdot)$ *in* $E_n(s)$ *which has the following properties.*

(i) $u_{n,s}$ *is a best approximation to* f_m *in* $\overline{E_n(s)}$.

(ii) $\operatorname{spect}(u_{n,s}) \subset [-1, 1]$.

(iii) *The error curve* $f_m - u_{n,s}$ *has a negative alternant of exact length* $2m+1$. *Moreover,* $u_{n,s}$ *has exactly* $2m-n$ *different characteristic numbers and each has an odd multiplicity.*

Proof. We prove Lemma 3.1 by induction on n. Since $S_{m,m+1} \subset \hat{S}_{m,m+1}$ we have already proved the case $n = m+1$ in Lemma 2.3. Assume that $m+1 < n \leqslant 2m-1$ and that the assertion of Lemma 3.1 is true for $S_{m,n-1}$. Let $s \in S_{m,n}$ and $a \in (0, \varepsilon_{n-1})$. Cancelling one of the $n-m$ negative components of s, we obtain $n-m$ different elements $s^{(k)}$, $1 \leqslant k \leqslant n-m$, of $S_{m,n-1}$. Without loss of generality we may assume that

$$\|f_m - u_{n-1,s^{(1)}}\| = \min_{1 \leqslant k \leqslant n-m} \|f_m - u_{n-1,s^{(k)}}\|. \tag{3.1}$$

The vector $s^{(1)}$ may be obtained from s by cancelling the j_0th component. Since $s \in S_{m,n}$ we have $1 < j_0 < n$. Hence, we can apply the standard construction to $u_{n-1,s^{(1)}}$ with an additional characteristic number between the $j_0 - 1$st and j_0th one. (Note that we now count the characteristic numbers according to their multiplicity.) Therefore, Theorem 1.4 yields the existence of a $v \in E_n(s)$ satisfying

$$\|f_m - v\| < \|f_m - u_{n-1,s^{(1)}}\|. \tag{3.2}$$

From Satz 2 in [8] it is known that there exists a best approximation $u_{n,s} = u_{n,s}(a; \cdot)$ to $f_m(a; \cdot)$ in $\overline{E_n(s)}$. The induction hypothesis, (3.2) and $ag_m \in \overline{E_{n-1}(s^{(1)})}$ imply

$$\|f_m - u_{n,s}\| < \|f_m - u_{n-1,s^{(1)}}\| \leqslant \|f_m - ag_m\|. \tag{3.3}$$

Since according to (2.6)

$$\partial E_n(s) \subset \bigcup_{k=1}^{n-m} \overline{E_{n-1}(s^{(k)})} \cup \{v \in E_{n-1} : \text{sign } (v)$$
$$\text{contains at most } m-1 \text{ positive components}\}$$

we conclude from (3.1), (3.3), the induction hypothesis and Lemma 1.3 that $u_{n,s} \in E_n(s)$. Hence, $u_{n,s}$ is an LBA. The proof of (ii) is completely analogous to the proof of Lemma 2.3(ii).

For the proof of (iii) we define for every $s \in S_{m,n}$

$$u_{n,s}(0; x) := 0, \qquad x \in I,$$

and

$$\phi : [0, \varepsilon_{n-1}) \to C(I),$$
$$\phi(a) := f_m(a; \cdot) - u_{n,s}(a; \cdot).$$

With the same arguments as in the proof of Lemma 2.3, we obtain the continuity of ϕ at $a = 0$ for every $s \in S_{m,n}$. Thus there is a number $\varepsilon_n \in (0, \varepsilon_{n-1})$ such that $f_m(a; \cdot) - u_{n,s}(a; \cdot)$ has at most an alternant of length $2m+1$ for every $a \in (0, \varepsilon_n)$ and $s \in S_{m,n}$ because this is true for $a = 0$. By the same argument, this alternant has a negative sign on the right if it has the exact length $2m+1$. From Theorem 6.1 in [2] we conclude that $u_{n,s}(a; \cdot)$ has at most $2m-n$ different characteristic numbers. Because of Definition 2.1(ii) we may split the sequence $s = (s_1, ..., s_n)$ into subsequences of the form $(+1, -1, +1, ..., -1, +1)$. The splitting is done between consecutive positive signs of s. Each subsequence has one more positive sign than negative ones. Hence, the splitting yields exactly $2m-n$ subsequences. Therefore, each $v \in E_n(s)$ has at least $2m-n$ different characteristic numbers. This completes the proof of Lemma 3.1. ∎

As a consequence we obtain the exponentially increasing lower bound for c_n.

THEOREM 3.2. *The maximal number of local best approximations in* E_n, $n \geqslant 4$, *is at least*

$$c_n \geqslant \max_{1 \leqslant i \leqslant [(n-1)/2]} \binom{n-1-i}{i} \geqslant \frac{2^{(2/3)n}}{5\sqrt{(2/3)n}}. \tag{3.4}$$

Proof. Let $n \geqslant 4$, $1 \leqslant i \leqslant [(n-1)/2]$ and $m := n - i$. Obviously $2 \leqslant m < n \leqslant 2m-1$. From the preceding lemma we have $c_n \geqslant \# S_{m,n}$.

A simple combinatorial argument, however, shows

$$\# S_{m,n} = \binom{m-1}{n-m} = \binom{n-1-i}{i}.$$

Here $i = n - m$. This proves the first inequality.

Now write $n - 1 = 3k + l$ with $k \in \mathbb{N}$ and $l \in \{0, 1, 2\}$. Then

$$c_n \geqslant \binom{n-1-k}{k} = \binom{2k+l}{k}.$$

From Stirling's formula we obtain

$$\binom{2k+l}{k} \geqslant \frac{2^{2k}}{\sqrt{2k\pi}} \delta_l,$$

where

$$\begin{aligned} \delta_l &= 1, \qquad l = 0, 1 \\ &= 2, \qquad l = 2. \end{aligned}$$

This implies the second inequality for c_n. ∎

For small values of n Theorem 2.5 gives a better lower bound for c_n than Theorem 3.2. The estimates of Theorem 3.2, however, show that c_n grows faster than any polynomial of n. In this context one has to notice that our technique can at best yield the estimate $c_n \geqslant 2^n$, because we identify the LBAs through their sign vector and cannot distinguish between different LBAs in the same sign class. Since the approximation of the function $\cos(\pi/2)\,x$ shows that a sign class may contain more than one LBA (cf. [4]), our lower bounds probably are not sharp.

4. The Existence of a Function with Six Local Best Approximations in E_4

The results of Section 2 yield the estimate

$$5 \leqslant c_4 \leqslant 9.$$

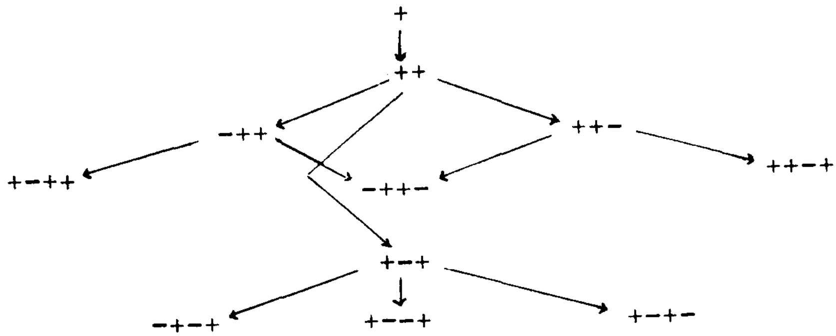
In this section we will improve this by constructing a function with at least six LBAs in E_4.

THEOREM 4.1. *Let $a > (3.5\pi)^2$. Then the function*

$$f(x) := \cos 3.5\pi x + a(\cosh x - 1), \qquad x \in I,$$

has at least six LBAs in E_4.

Proof. The following diagram shows the sign classes containing an LBA to f in E_n, $1 \leqslant n \leqslant 4$. The connecting arrows refer to the standard construction. The position in the diagram indicates the levels of the different LBAs.



Sign classes containing LBAs to $\cos 3.5\pi x + a(\cosh x - 1)$ in E_n, $1 \leqslant n \leqslant 4$.

Obviously, we have

$$\begin{aligned} f'(x) &= -3.5\pi \sin 3.5\pi x + a \sinh x \\ &\geqslant -(3.5\pi)^2 x + ax \geqslant 0, \qquad x \geqslant 0. \end{aligned}$$

Since f is an even function this implies

$$\begin{aligned} \min_{x \in I} f(x) &= f(0) = 1, \\ \max_{x \in I} f(x) &= f(-1) = f(1). \end{aligned} \tag{4.1}$$

Therefore, the BA to f in E_1 is

$$u^{(1)}(x) := \tfrac{1}{2}\{f(0) + f(1)\}, \qquad x \in I.$$

Because of (4.1) the error curve $f - u^{(1)}$ has a positive alternant of exact length three.

By Korollar 3 in [8] and Theorem 5.2 in [2], f has a unique BA $u^{(2)} \in E_2^+ \backslash E_1$ in E_2. Since f is an even function $u^{(2)}$ has the representation

$$u^{(2)}(x) = A \cosh Tx, \qquad x \in I,$$

with $A > 0$, $T > 0$.

By construction $a \cosh x - a \in E_3(+-+)$ is the unique BA to f in E_3. From Lemma 1.3 and Theorem 6.1 in [2], we conclude that $f - u^{(2)}$ has a negative alternant of exact length five. Hence, we can apply the standard construction to $u^{(2)}$. By standard arguments we then obtain the existence of three LBAs $u_1^{(3)} \in E_3(-++)$, $u_2^{(3)} \in E_3(+-+)$ and $u_3^{(3)} \in E_3(++-)$ in E_3 with

$$\|f-u_i^{(3)}\| < \|f-u^{(2)}\|, \qquad i=1,2,3. \tag{4.2}$$

Since $c_3 = 3$ (cf. [6]) there are no other LBAs to f in E_3 and each $u_i^{(3)}$, $i=1$, 2, 3, is a BA to f in its sign class. Hence, $u_2^{(3)}$ is the unique BA given above. Moreover, we obtain from the symmetry of f that

$$\|f-u_1^{(3)}\| = \|f-u_3^{(3)}\|. \tag{4.3}$$

Since $u_2^{(3)}$ satisfies the assumptions of Theorem 1.4 with $\sigma=-1$, we can apply the standard construction to $u_2^{(3)}$. Recalling that $u_2^{(3)}$ is the unique global solution in E_3, we conclude the existence of at least one LBA to f in each of the three sign classes

$$E_4(-+-+), \qquad E_4(+--+), \qquad E_4(+-+-).$$

Now we turn our attention to the LBAs $u_1^{(3)}$ and $u_3^{(3)}$. The same arguments as in the proof of Lemma 2.4 yield that $u_1^{(3)}$ and $u_3^{(3)}$ have an alternant of minimal length and satisfy the assumptions of Theorem 1.4 with $\sigma=-1$ and $\sigma=+1$, resp.

Let us apply the standard construction to $u_1^{(3)}$ with an additional characteristic number greater than the greatest one of $u_1^{(3)}$. We obtain a connected component C of the level set $\{v \in E_4 : \|f-v\| < \|f-u_1^{(3)}\|\}$ which intersects $E_4(-++-)$. From (4.2), (4.3) and Lemma 1.3 we conclude

$$\begin{aligned} C \cap \partial E_4(-++-) &= C \cap (E_3(++-) \cup E_3(-+-) \cup E_3(-++)) \\ &= \varnothing \end{aligned}$$

and hence by the connectedness $C \cap E_3 = \varnothing$. This implies the existence of at least one LBA to f in $E_4(-++-)$.

Now we apply the standard construction to $u_1^{(3)}$ with an additional characteristic number smaller than $\min\{t_0, -T\}$, where t_0 is the smallest characteristic number of $u_1^{(3)}$. By Theorem 1.4 we obtain a connected component C of the level set $\{v \in E_4 : \|f-v\| < \|f-u_1^{(3)}\|\}$ which intersects the set A of the elements of $E_4(+-++)$ with smallest characteristic number smaller than $-T$. If we replace $f_m(a;\cdot)$ by f, ag_m by $u^{(2)}$ and -1 by $-T$ in the first part of the proof of Lemma 2.4, we conclude with the same arguments that

$$C \subset A \subset E_4 \backslash E_3.$$

This implies the existence of an LBA u to f in $E_4(+-++)$. Since f is an even function, $u(-x) \in E_4(++-+)$ is also an LBA to f in E_4. This completes the proof. ∎

5. The Existence of a Function with Three Global Best Approximations in E_4

In this section we establish the following:

THEOREM 5.1. *There is a continuous function with three global best approximations in* E_3.

Proof. We consider the functions

$$f(a; x) := -\cos 2.5\pi x + 2a \cosh x, \qquad x \in I,\ a > 0.$$

Since $\hat{S}_{2,3}$ consists of the three sign vectors

$$s_1 = (-1, +1, +1), \qquad s_2 = (+1, -1, +1), \qquad s_3 = (+1, +1, -1),$$

we know from Lemma 2.3 that each sign class $E_3(s_i)$, $i = 1, 2, 3$, contains an LBA $u_i(a; \cdot)$, $a > 0$, to $f(a; \cdot)$ in E_3. Moreover, each $u_i(a; \cdot)$ is a BA to $f(a; \cdot)$ in $\overline{E_3(s_i)}$. Since $c_3 = 3$ there are no other LBAs to $f(a; \cdot)$ in E_3. The symmetry of $f(a; \cdot)$ and the definition of the sign classes, imply

$$\|f(a; \cdot) - u_1(a; \cdot)\| = \|f(a; \cdot) - u_3(a; \cdot)\|, \qquad a > 0. \tag{5.1}$$

Furthermore, $u_2(a; \cdot)$ has either the representation

$$u_2(a; x) = A \cosh Tx - B, \qquad x \in I, \tag{5.2}$$

or

$$u_2(a; x) = \alpha + \beta x^2, \qquad x \in I, \tag{5.2$'$}$$

where A, B, T, α, β depend on a. First we prove that

$$\|f(a_0; \cdot) - u_2(a_0; \cdot)\| < \|f(a_0; \cdot) - u_1(a_0; \cdot)\| \tag{5.3}$$

for some $a_0 > 0$. Let h be the BA to $g(x) := -\cos 2.5\pi x$ in the Haar space $V := \operatorname{span}\{e^{-x}, 1, x, x^2, x^3, e^x\}$. Since g is even and has a negative alternant of exact length five, h has the representation

$$h(x) = -2a_0 \cosh x + \alpha + \beta x^2$$

with $a_0 > 0$, $\beta > 0$. Hence

$$g(x) - h(x) = f(a_0; x) - \alpha - \beta x^2$$

has an alternant of length seven and $\alpha + \beta x^2 \in E_3(+-+)$ is the unique BA to $f(a_0; \cdot)$ in E_3. This yields (5.3).

Next we prove

$$\|f(2; \cdot) - u_1(2; \cdot)\| < \|f(2; \cdot) - u_2(2; \cdot)\|. \tag{5.4}$$

We have computed the LBAs to $f(2; \cdot)$ in E_3 on the discrete set $X = \{-1 + (i-1)/100 : 1 \leqslant i \leqslant 201\}$. The numerical calculation yields

$$v_1(x) = (a + bx)\, e^{tx} + c e^{sx}, \qquad v_3(x) = v_1(-x),$$
$$v_2(x) = \alpha + \beta x^2$$

with

$$a = 0.899536, \qquad b = 0.706562, \qquad c = 3.070845,$$
$$t = -2.927102, \qquad s = 0.652912,$$
$$\alpha = 3.993427, \qquad \beta = 2.108968$$

and

$$\max_{x \in X} |f(2; x) - v_1(x)| = 0.970381,$$

$$\max_{x \in X} |f(2; x) - v_2(x)| = 0.993427.$$

By linear interpolation of $f(2; \cdot) - v_1$ we obtain

$$\|f(2; \cdot) - u_1(2; \cdot)\| \leqslant \|f(2; \cdot) - v_1(\cdot)\| \leqslant 0.9715. \tag{5.5}$$

Now, let $y_i := -0.8 + (i-1)\,0.4$, $1 \leqslant i \leqslant 5$. Considering round-off errors, we get

$$(-1)^i \{f(2; y_i) - v_2(y_i)\} \geqslant 0.9934, \qquad 1 \leqslant i \leqslant 5.$$

Since $u_2(2; \cdot)$ has the representation (5.2) or (5.2′), this implies

$$\|f(2; \cdot) - u_2(2; \cdot)\| \geqslant 0.9934. \tag{5.6}$$

Inequalities (5.5) and (5.6), however, prove (5.4). Now the map

$$(0, \infty) \ni a \to \|f(a; \cdot) - u_1(a; \cdot)\| \\ - \|f(a; \cdot) - u_2(a; \cdot)\| \tag{5.7}$$

is continuous by Theorem 12.5 in [3], since $f(a; \cdot)$ has exactly the three LBAs $u_i(a; \cdot)$, $i = 1, 2, 3$, in E_3 for every $a > 0$. Since the function given in (5.7) has a zero between $a = a_0$ and $a = 2$, the proof is complete. ∎

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